IWR – Universität Heidelberg
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Due date: **25.10.2013**

Homework No. 1
Numerical Methods for PDE, Winter 2013/14

Problem 1.1: Variational equations in $\mathbb{R}^n$

Given a symmetric, positive definite matrix $A \in \mathbb{R}^{n\times n}$ and a vector $b \in \mathbb{R}^n$ and the "energy functional"

$$E(x) = \tfrac{1}{2} x^T A x - x^T b, \tag{1.1}$$

(a) Derive the variational equation of the minimization problem by studying the derivative of the auxiliary function $\Phi(t) = E(x + ty)$ for arbitrary $y \in \mathbb{R}^n$.

(b) Show that a vector $x \in \mathbb{R}^n$ minimizes $E(x)$, that is,

$$E(x) \le E(y) \quad \forall y \in \mathbb{R}^n,$$

if and only if

$$Ax = b.$$

(c) Conclude that the minimizer x exists and is unique.

Problem 1.2: Minimizing sequence

(a) Show that a sequence $\{x^{(k)}\}$ such that for the energy functional in (1.1) holds

$$E(x^{(k)}) \to \inf_{y\in\mathbb{R}^d} E(y), \tag{1.2}$$

necessarily converges to the minimizer x from Problem 1.1. The "binomial formula" $x^T Ax - y^T Ay = (x+y)^T A(x-y)$ and the fact that A is invertible are useful ingredients to this proof.

(b) Show without assuming the existence of the minimizer x, that a sequence $\{x^{(k)}\}$, for which (1.2) holds is necessarily a Cauchy sequence. Can you conclude the existence of a minimizer x?

Problem 1.3: Integration by parts

Let Ω be a domain in $\mathbb{R}^d$. Use the Gauß theorem for smooth vector fields $\varphi : \Omega \to \mathbb{R}^d$, namely,

$$\int_\Omega \nabla\cdot\varphi \,\mathrm{d}x = \int_{\partial\Omega} \varphi \cdot \mathbf{n} \,\mathrm{d}s,$$

to show Green's first and second formula (for smooth scalar functions u and v)

$$-\int_\Omega \Delta u v \,\mathrm{d}x = \int_\Omega \nabla u \cdot \nabla \,\mathrm{d}x - \int_{\partial\Omega} \partial_n u v \,\mathrm{d}s$$

$$\int_\Omega (u\Delta v - v\Delta u)\,\mathrm{d}x = \int_{\partial\Omega} (u\partial_n v - v\partial_n u)\,\mathrm{d}s.$$

Here, $\mathbf{n}$ is the outward unit normal vector to Ω on $\partial\Omega$. The differential operators have the meaning:

$\nabla u = (\partial_1 u, \dots, \partial_d u)^T$	gradient
$\partial_n u = \mathbf{n} \cdot \nabla u$	normal derivative
$\nabla\cdot\varphi = \partial_1\varphi_1 + \cdots + \partial_d\varphi_d$	divergence
$\Delta u = \nabla\cdot\nabla u = \partial_{11} u + \cdots + \partial_{dd} u$	Laplacian

Each problem 4 points.