IWR – Universität Heidelberg
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Homework No. 4
Numerical Methods for PDE, Winter 2013/14

Problem 4.1: Convection-Diffusion-Reaction Equation

Given is the problem

$$\begin{aligned} Lu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ \partial_n u &= 0 \quad \text{on } \Gamma_N. \end{aligned}$$

on a domain $\Omega \subset \mathbb{R}^d$ with sufficiently smooth boundary, which is divided in a Dirichlet boundary part Γ_D and a Neumann boundary part Γ_N.

The operator

$$Lu(x) = -\nabla \cdot (a \nabla u(x)) + \beta(x) \cdot u(x) + r(x)u(x)$$

has sufficiently smooth data functions $\beta(x)$ and $r(x)$ and the parameter a is positive.

(a) Deduce the variational formulation from the classical formulation above for the function space

$$V = H^1_{\Gamma_D}(\Omega) := \{\varphi \in H^1(\Omega) : \varphi|_{\Gamma_D} = 0\}.$$

(b) Prove unique existence of a solution in case the following three assumptions hold:

- $\Gamma_D = \partial\Omega$ (the whole boundary has homogeneous Dirichlet data),
- $\nabla \cdot \beta = 0$ (solenoidal transport field),
- $r(x) \geq r > 0$.

(c) **Bonus (2 points):** Discuss the unique solvability in the original setting. Do we need conditions on $\beta(x)$ and $r(x)$? Try to find a more general condition for $\beta(x)$ and $r(x)$ than in (b).

Hint: You can use the generalized Poincaré's inequality

$$\|u\| \leq c_P \|\nabla u\|$$

for functions in $H^1_{\Gamma_D}(\Omega)$ where Γ_D has positive boundary measure.

Problem 4.2: Integral node functionals

A finite element on a triangle shall consist of the space of quadratic polynomials P_2 and shall utilize the node functionals $\mathcal{N}_i$ defined by

$$\begin{aligned} \mathcal{N}_i(f) &= f(p^i) & i &= 1,2,3, \\ \mathcal{N}_i(f) &= \frac{1}{|E_{i-3}|} \int_{E_{i-3}} f(x)\,\mathrm{d}s, & i &= 4,5,6. \end{aligned}$$

Here, E_i is the edge of the triangle facing the vertex p^i, and $|E_i|$ is its measure.

(a) Show that this element is unisolvent.

(b) Derive a basis $\{\varphi_j\}$ for P_2 such that $\mathcal{N}_i(\varphi_j) = \delta_{ij}$.

Problem 4.3: Unisolvence

The bilinear quadrilateral element was introduced in class as a parametric element. Here we want to study the nonparametric version. Let $Q_1 = \operatorname{span}\{1, x, y, xy\}$. Let the cell K be the square with corners

$$p^1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, p^2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, p^3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, p^4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The node values $\mathcal{N}_i$ of the finite element are the function values in the corners.

(a) Show that the element defined this way is not unisolvent.

(b) Why is the parametric element unisolvent on K?

(c) Sketch the shape functions of the parametric Q_1 element on the cell K.

Each problem 4 points.