IWR – Universität Heidelberg
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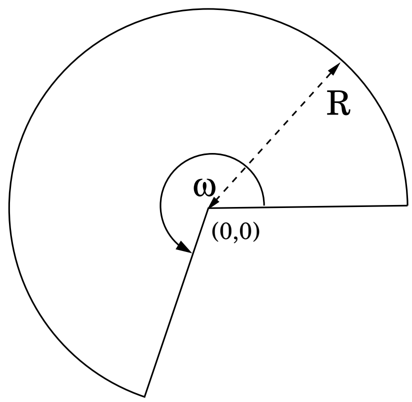
Due date: **6.12.2013**

# Homework No. 6
# Numerical Methods for PDE, Winter 2013/14

## Problem 6.1: Corner singularity (10 points)

Let the domain $\Omega \subset \mathbb{R}^2$ be the sector




with radius $R = 1$ and interior angle $\omega$. In polar coordinates, this domain is described by $r \in (0, 1)$ and $\vartheta \in (0, \omega)$.

**(a)** Verify: The Laplace equation in polar coordinates is

$$-\frac{\partial^2}{\partial r^2} u(r, \vartheta) - \frac{1}{r}\frac{\partial}{\partial r} u(r, \vartheta) - \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2} u(r, \vartheta) = f(r, \vartheta).$$

**(b)** Verify that the function

$$u(r, \vartheta) = r^{\frac{\pi}{\omega}} \sin\left(\tfrac{\pi}{\omega}\vartheta\right)$$

solves the Laplace equation with zero boundary values on the legs of the angle and smooth boundary values $\sin\left(\frac{\pi}{\omega}\vartheta\right)$ on the circumference.

**(c)** Show that $u \notin W^{2,2}(\Omega)$ if $\omega > \pi$. **Hint:** It is sufficient to consider the derivative $\partial_{rr} u$.

**(d)** Show that on a triangle of size $h$ containing the origin, this function cannot be approximated by linear functions better than

$$|u - u_h|_1 \lesssim h^{\frac{\pi}{\omega}}.$$

Here, the operator "$\lesssim$" means: there is a positive constant $c$ independent of $h$ (but in this case depending on $u$) such that $|u - u_h|_1 \leq c h^{\frac{\pi}{\omega}}$

**Problem 6.2: Meshes (4 points)** Draw a shape regular mesh into the following domain: