IWR – Universität Heidelberg
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Due date: **discussion in first tutorials in January**

Homework No. 9
Numerical Methods for PDE, Winter 2013/14

Problem 9.1: Short questions

(a) Consider a function $u \in L^2(\Omega)$. How can you formulate the weak Laplacian for this function?

(b) Formulate the Riesz representation theorem and the Lax-Milgram lemma.

(c) In which sense is the Lax-Milgram lemma an enhancement of the Riesz representation theorem?

(d) What does V-elliptic mean for a bilinear form $a(u, v)$ with u, v in V?

(e) Is the bilinear form $a(u, v) = (\nabla u, \nabla v)$ V-elliptic for $V = H^1(\Omega)$?

(f) Explain what is meant by the term 'Galerkin orthogonality'.

(g) Formulate Ceá's lemma and state the requirements for this result.

(h) Formulate a linear error functional $J(\varphi)$ which represents the L^2-Norm for $\varphi = u - u_h$.

(i) Describe the duality argument (Aubin-Nitsche trick) for error estiamtes in 'weak' norms. What is it used for?

(j) Describe the concept of parametric finite elements.

(k) How do node values induce continuity into a finite element space?

(l) What does the term unisolvence mean for a polynomial ansatz space?

(m) When is a triangulation called shape regular?

(n) Write a typical a priori error estimate.

(o) State the Bramble-Hilbert Lemma.

(p) Describe the connection between error estimates, transformation of the reference cell and the Bramble-Hilbert Lemma.

(q) Which order of convergence can we obtain for the L^2- and the H^1-norm of an approximation with quadratic finite elements in the best case?

(r) What is error pollution?

(s) What is the difference between a-priori and a-posteriori error estimates?

(t) Describe the concept of a dual weighted error estimator.

Problem 9.2: Energy Norm

Show that

$$a\,(u, v) := \int_\Omega \nabla u(x) \cdot \nabla v(x)\; dx, \qquad\qquad |u|_{H^1(\Omega)_0} := \left(|\nabla u(x)|^2\; dx\right)^{\frac{1}{2}},$$

define a scalar product and a norm for the space $H^1_0(\Omega)$.

Problem 9.3: H^1 Regularity

Consider the domain $\Omega = (0,1)^2$. Is the function

$$u(x,y) = \sin\left(\ln\left(\frac{1}{r}\right)\right), \qquad \text{with} \qquad r = \left(x^2 + y^2\right)^{\frac{1}{2}}$$

in $H^1(\Omega)$?

Problem 9.4: Sobolev Spaces

Which of the following estimates

$$\begin{array}{llll}
\text{a)} & \|u\|_{L^\infty(\Omega)} \leq c\|u\|_{W^{2,2}(\Omega)}, & u \in W^{2,2}(\Omega),\ \Omega \subset \mathbb{R}^3, \\
\text{b)} & \|u\|_{L^\infty(\Omega)} \leq c\|u\|_{W^{1,1}(\Omega)}, & u \in W^{1,1}(\Omega),\ \Omega \subset \mathbb{R}^1, \\
\text{c)} & \|u\|_{L^\infty(\Omega)} \leq c\|u\|_{W^{1,2}(\Omega)}, & u \in W^{1,2}(\Omega),\ \Omega \subset \mathbb{R}^2, \\
\text{d)} & \|u\|_{L^1(\partial\Omega)} \leq c\|u\|_{W^{1,1}(\Omega)}, & u \in W^{1,1}(\Omega),\ \Omega \subset \mathbb{R}^2,
\end{array}$$

are valid? Describe also the meaning of the above mentioned norms and function spaces.

Problem 9.5: Convection-Diffusion Equation

Consider the following convection-diffusion equation

$$\begin{aligned}
-\Delta u(x,y) + 3\partial_y u(x,y) &= f(x,y) && \text{in } \Omega, \\
u(x,y) &= 0 && \text{on } \partial\Omega,
\end{aligned}$$

on a convex polygon $\Omega \subset \mathbb{R}^2$.

- **(a)** Prove the existence of a weak solution $u \in H_0^1(\Omega)$ for $f \in L^2(\Omega)$. Show that this solution is also unique.
- **(b)** Formulate the Galerkin approximation and prove the existence of a unique discrete solution u_h in the conforming finite dimensional space V_h.
- **(c)** Prove that the Galerkin approximations u_h converge to the solution of the continuous problem u for $h \to 0$.
- **(d)** Formulate the dual problem for an appropriate linear error functional $J(\cdot)$ representing the H^1-norm.
- **(e)** Discuss the solution theory for the dual problem.